# Startup Growth and Venture Returns

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We use AngelList data from thousands of early-stage venture investments to solve for how each successive year of a startup's existence affects investment returns. This allows us to create a model of how quickly winning venture investments grow and how that rate of growth decays. Our model shows that at the seed stage investors would increase their expected return by broadly indexing into *every* credible deal, a finding that does not hold at later stages. Our results also suggest that startups staying private longer have created a powerful engine for unbounded wealth creation entirely outside the public markets; we conclude with an argument rooted in social equity for why retail investors should have access to a broad-based index of early-stage venture investments.

## Introduction

Venture capital has largely resisted the quantification that has revolutionized modern finance. In lieu of mathematical modeling, venture capitalists tend to subscribe to pieces of folk wisdom around their investing activities. How, when, and in what a VC invests all tend to have only the thinnest veneer of theoretical or empirical justification.

In this paper we use AngelList's broad database of early-stage investments to create and fit a model of venture capital investment returns from first principles. Fundamentally, we rely on two concepts. The first is that startups tend to grow faster in their earliest years, for which we are able to provide empirical support from AngelList data. The second is that early-stage investments have longer durations than later-stage investments in those same companies, which is tautological. Taken together, the two concepts imply that winning early-stage investments have more years to compound at higher rates of growth.

While our high-level model is relatively uncontroversial, there are a number of provocative conclusions that arise from fitting it to the AngelList data. Our results suggest that after five years a winning seed-stage investment begins to draw its return multiple distribution from an $\alpha < 2$ (i.e., unbounded mean) power law. This means that the regret an investor could have for missing a winning seed-stage investment is theoretically infinite, a phenomenon that does not appear to hold for later-stage investments. The implication is that investors increase their expected return by indexing as broadly as possible at the seed stage (i.e., by putting money into every credible deal), because any selective policy for seed-stage investing—absent perfect foresight—will eventually be outperformed by an indexing approach.

Our model suggests that investors in different rounds of fundraising experience different distributions of expected returns for their winning investments, and that these distributions change so much over a startup's lifetime that early- and late-stage venture capital should be treated as distinct asset classes. (This is perhaps surprising because these investors hold identical—albeit time-shifted—assets.) We believe this provides the first quantitative justification for why the primary divide in venture capital is between early- and late-stage investors, as opposed to by sector.

The first several sections of this paper assume familiarity with the math that underlies modern data science: calculus, probability, statistics, and optimization, as well as some scattered insights from the finance literature. Readers with less technical backgrounds are encouraged to skip ahead to the Discussion section at the end of the paper while perusing the self-contained plots and graphs along the way.

# Background

In this section we provide definitions of our core concepts, introduce power-law distributions and an existing academic model of venture capital returns, and discuss the AngelList dataset that we will use in our empirical results.

### Preliminary Definitions

Subtle issues around investment duration motivate precise definitions of our core concepts.

**Definition 1.** *An* investment *is a series of timestamped cashflows* $\{(v_1, t_1), \ldots, (v_n, t_n), (v_r, t_{now})\}$ *where* $v_i < 0$ *indicates money paid in (invested),* $v_i > 0$ *indicates money paid out (distributed), and* $t_i$ *is the timestamp. If* *the investment is still active, the value $v_r$ with a current timestamp $t_{now}$ is appended representing the investment's* residual value.

In general, most of the value of an early-stage venture capital investment that has not exited (through a merger, acquisition, or IPO) will be represented in its residual value.

**Definition 2.** *An investment's* Internal Rate of Return (IRR) *is the rate of growth $r$ that equilibrates between incoming and outgoing cashflows.*

$$\sum_{v>0} v \cdot (1+r)^t = \sum_{v<0} |v| \cdot (1+r)^t$$

In this paper we will quote IRRs on an annualized basis. For overall clarity, we will often abuse language and refer to the quantity $1+r$ as the "IRR" of the investment, particularly in the case of draws from a power-law distribution.

**Definition 3.** *An investment's* return multiple $m$ *is the combined sum of the distributed and residual values of the investment divided by the total amount invested. Formally:*

$$m \equiv \frac{\sum_{v>0} v}{\sum_{v<0} |v|}$$

Often when an investment exits, part of the money is held back in an escrow account, while the bulk is paid out immediately. We want to be able to handle situations like this in an intelligent way. Consequently, throughout this work we will always be referring to the formal notion of an investment's *effective duration* whenever we talk informally about an investment's "duration".

**Definition 4.** *An investment's* effective duration *is defined as the amount of time such that the investment's IRR implies its return multiple. Formally, $d$ is an investment's effective duration that solves:*

$$m = (1+r)^d$$

*Observe that this equation has multiple solutions when $m = 1 + r = 1$ exactly. For completeness, in that case we would treat an investment's effective duration as the money-weighted average investment duration, i.e., the same result that we would get if we had added a small $\epsilon > 0$ to the investment's final cashflow.*

For much of this paper we only consider investments that have not lost value. We dub these *winning* investments.

**Definition 5.** *A* winning *investment has $1 + r \geq 1$ (or equivalently, $m \geq 1$).*

### An Introduction to Power Laws

The seminal quantitative work on modern power laws is Clauset et al. (2009). Taleb (2001, 2007) also discusses power laws at length from a less quantitative perspective, focusing on their relationship to finance, culture, and society.

**Definition 6.** *A power-law distribution (with shape parameter $\alpha > 1$) is distributed according to the probability density function*

$$f(x) \equiv \frac{\alpha - 1}{x_{min}} \left( \frac{x}{x_{min}} \right)^{-\alpha}$$

*for $x \geq x_{min}$.*

In this paper we will only consider power-law distributions where $x_{min} = 1$, reducing the power law PDF to simply:

$$f(x) = (\alpha - 1)x^{-\alpha}$$

There are three cases to consider depending on the shape parameter $\alpha$, each of which can have different implications for portfolio construction and assessment:

- When $\alpha > 3$, the distribution has finite mean and finite variance. With finite variance, the Central Limit Theorem holds and so important portfolio theory concepts like the Sharpe Ratio have meaning. If investments draw their returns from a distribution with $\alpha > 3$ and investors select randomly among them, then the number of investments in a portfolio does not affect that portfolio's expected mean or expected median return.
- When $2 < \alpha \leq 3$, the distribution has finite mean but unbounded (infinite) variance; as a result, the Central Limit Theorem does not hold. If investments draw their returns from such a distribution, making more investments at random will increase a portfolio's expected median, but not expected mean, return.
- When $\alpha \leq 2$, both the mean and variance are unbounded. If investments draw their returns from such a distribution, making more investments at random will increase *both* a portfolio's expected median and expected mean return.

This last case of an "escaping" distribution is extremely difficult to make intelligible. Such distributions are far outside the realm of our typical experience. In our opinion, the most illuminating description of such a distribution comes from Taleb (2001), who describes a "Refugee Probability Distribution": for each additional day that a refugee spends outside of their homeland, the number of days that they can expect to wait to return increases by more than one.

The heavy tails of power laws are generally thought of as their most unintuitive property. However, another unintuitive property is that the power-law distribution, but *unlike* a Gaussian or uniform distribution, has qualitative properties (i.e., the existence of moments) that are highly dependent on its shape parameters.

## An Existing Model of Venture Capital

Metrick and Yasuda (2010a) created a model of venture capital returns that they also used in their popular textbook (Metrick and Yasuda, 2010b). This model was also used as the foundation of a recent, widely cited study on the overvaluation of so-called "unicorn" companies (Gornall and Strebulaev, 2019). What we call the "Standard Model" of academic venture capital appears to be based on observations of later-stage near-public companies and it works as follows:

- Venture capital investments obey the dynamics of Black and Scholes (1973) and Merton (1973): The value of each investment follows a heat-diffusion process where its log change in any time period is drawn from a volatile normal distribution (and therefore its future value is drawn from a lognormal distribution).
- Investments exit according to an independent draw from an exponential distribution.

It is well known by investors that the return multiple of winning early-stage venture capital investments is consistent with a power-law distribution. Any credible descriptive model of the nature of venture capital returns must be able to produce these heavy-tailed return multiples. The Standard Model can produce returns that appear to be drawn from heavy-tailed power laws through the extension of a winning investment's duration for exponentially long periods of time. (This phenomenon is discussed in passing in Metrick and Yasuda (2010b).)

For our purposes, there are two testable predictions that the Standard Model makes. The first is that the IRRs of venture capital investments should have light tails. The second is that investment duration and IRR should be uncorrelated. We proceed to introduce the proprietary, curated database of early-stage venture capital investments that we will use to refute both of these predictions.

### The AngelList Dataset

Since 2013, AngelList has syndicated more than three thousand investments into startup companies. The AngelList data on these investments includes not just valuations, which are often publicly announced, but price-per-share figures, which are typically closely held. This is an important and often overlooked point: increases in headline valuations determine investor returns only inasmuch as they result in changes in share price. To assess residual values we follow the standard venture capital practice of marking all investments to the price-per-share of the most recent funding round.

AngelList syndicates have an atypical cost structure for venture capital that is especially good for assessing the performance of individual investments. While the typical VC fund charges a management fee as well as carried interest on a net portfolio basis, AngelList typically charges a small setup fee for each syndicated investment along with 20% carried interest (i.e., 20% of the positive return) if it makes money.[1] All of the investment results we present in this paper are net of fees and carry, which carry may be unrealized if the investment has not exited. In the following sections we will only consider winning investments, as the returns of money-losing venture investments do not appear to follow a power law. We will revisit the issue of money-losing investments in the Discussion section.

AngelList's cost structure allows us to look at returns net of fees in a straightforward way on the level of individual investments. Unfortunately, it poses some challenges when examining the most recently made investments. Because of setup fees, all AngelList investments begin their life underwater. However, a certain fraction of investments run counter to this: *bridge rounds* immediately before a priced round that are sold at a discount to that close-in round's price. For instance, a startup founder may circulate a small seed round to friends, family, and advisors at a 20% discount to their first equity financing having already started discussions with a venture capitalist about a priced equity round, which equity round may happen soon thereafter.

[1] Some AngelList syndicates charge less than 20% carry and a small number of AngelList syndicates charge management fees. We uniformly impose 20% carry for each syndicate and any difference in fee structure is not material to our analysis.

Consequently, when we look at non-negative investments with the shortest investment durations these bridge rounds result in a serious skew towards short-duration investments looking unrealistically attractive. For instance, in our exploratory data analysis it was typical to see some annualized IRRs higher than 4,000% from these bridge instruments (e.g., from an investment that gets marked up 25% three weeks after it was made). In our judgement, these investments represent tactical, temporal anomalies rather than real growth in company valuations. Supporting this contention, investors in these bridge rounds typically will not have the ability to sell their investment at a profit at the close-in financing.

To allay concerns about the temporal anomalies that these bridge notes induce, we filter our dataset down to only those investments that have had at least one year to season. This filtering has the effect of removing the close-in bridge notes. Observe that we do not exclude these bridge notes entirely as they are real early-stage investments and by definition often the earliest round of financing a company raises, but only consider them once they have been around for at least a year. The only exception we make to this seasoning rule is for companies that exit within a year of the investment; in this case we view the exit as an external validation of realized growth as the bridge-round investors will sell their holdings at exit. Our "seasoning filter" of one year is in line with past research into venture capital; Korteweg and Sorensen (2011), using a large historical database of VC investments, suggest a median of about ten months and a mean of about 13 months between financing rounds of a company.

We add one additional filter to our dataset.[2] For reasons that will become clear, we are extremely interested in knowing *when* an investment is made in a company relative to its first investable opportunity. In order to know this value we must be reasonably sure of the company's first investable opportunity. Consequently, we filter our dataset down to only those companies that raised a seed round with AngelList, and those companies' subsequent rounds of fundraising that we have information on. Taken together, these three filters of (1) winning syndicates that (2) have had at least one year to season (or have exited) and (3) for which we know the date of the seed round, leaves us with a database of 684 non-negative investments, constituting 487 initial seed investments and 197 later follow-on investments into those the same companies. While this is a large dataset from the perspective of venture capital, it is somewhat smaller than the rules of thumb around power law estimation. Clauset et al. (2009) suggests needing at least 1,000 datapoints to confirm a power-law distribution. This sparseness motivates us to be more proactive about controlling overfitting when we perform parameter estimation later in this work.

[2] For completeness and where appropriate we will also discuss summary statistics for the entire dataset of 1,070 winning AngelList syndicates, regardless of duration or stage, on the platform.

**Table 1.** Summary statistics from the AngelList dataset of 684 non-negative investments that we consider in this paper.

| | IRR | Multiple | Effective Duration |
|---|---|---|---|
| Minimum | +0% | 1.0x | 0.4 |
| Median | +21% | 1.7x | 3.0 |
| Mean | +35% | 2.7x | 3.1 |
| Maximum | +520% | 115x | 6.4 |

IRRs are in percentages per year (i.e., "p.a."). Effective durations are in years.

## Theoretical Model Building from Empirical Results

Figure 1 shows the power law fit for the return multiples of winning investments.

As expected, return multiples are consistent with a fairly extreme power-law distribution; the fit here has $\alpha = 2.42$. This fit was made using the python `powerlaw` package using the log-likelihood maximization methods derived originally from the work of Clauset et al. (2009). The fit on the complete dataset of 1,070 winning AngelList investments is similar at $\alpha = 2.46$.

Our first novel empirical contribution is the finding that IRRs are *also* consistent with a power-law distribution. Figure 2 shows the distribution of IRRs of winning AngelList investments with $1 + r \geq 1$.

The fit here has $\alpha = 4.8$, a relatively tame power law. The fit on the complete dataset of 1,070 winning AngelList syndicates is smaller at $\alpha = 4.6$, reflecting the more extreme IRR values of the close-in bridge financings that are excluded from the dataset by our seasoning filter.

These results motivate us to consider an alternative to the Standard Model in which each winning investment has its IRR $1 + r$ drawn i.i.d. from a power-law distribution with shape parameter $A$. Recall that in the Standard Model, investments needed to potentially persist for exponentially long in order to produce a heavy-tailed distribution of return multiples. When IRRs themselves are drawn from a power-law distribution, the

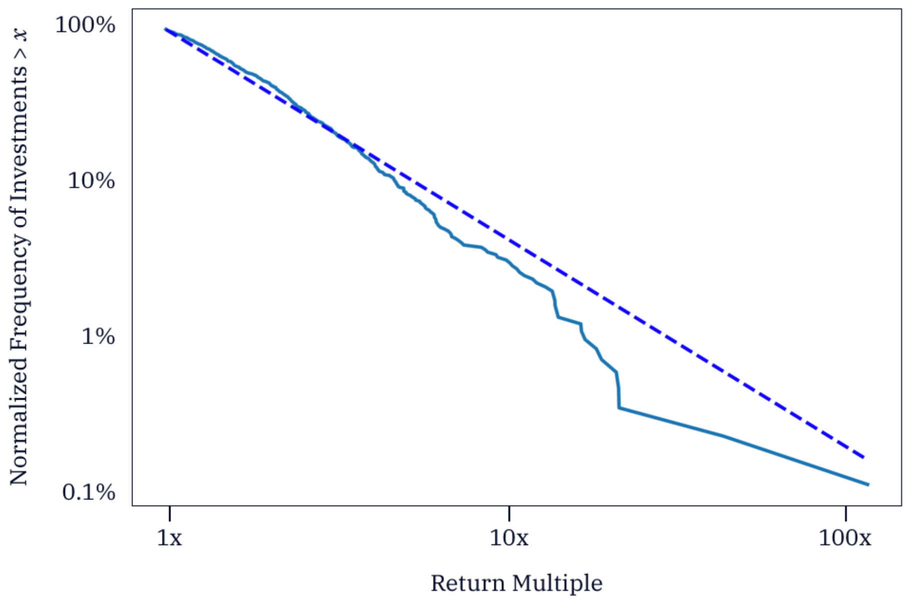


**Fig. 1.** As expected, the distribution of return multiples of winning investments in our dataset (solid blue line) is consistent with a fairly extreme power-law distribution (dotted blue line).

creation of heavy-tailed return multiples is straightforward. The following result, proven in an appendix, links the distribution of return multiples to the distribution of IRRs through effective durations.

**Theorem 1.** *An investment with its IRR $1 + r$ drawn from a power-law distribution with shape parameter $A$ and an effective duration of $t$ years has its return multiple correspondingly drawn from a power-law distribution with shape parameter*

$$\alpha \equiv \frac{A + t - 1}{t}$$

Observe that when $t = 1$ (so that all investments last exactly one year), we get that $\alpha = A$ as we would expect. Observe also that when $t > 1$ we get $\alpha < A$, and that as $t \to \infty$ we get that $\alpha \to 1$, the theoretical limit of the distribution.

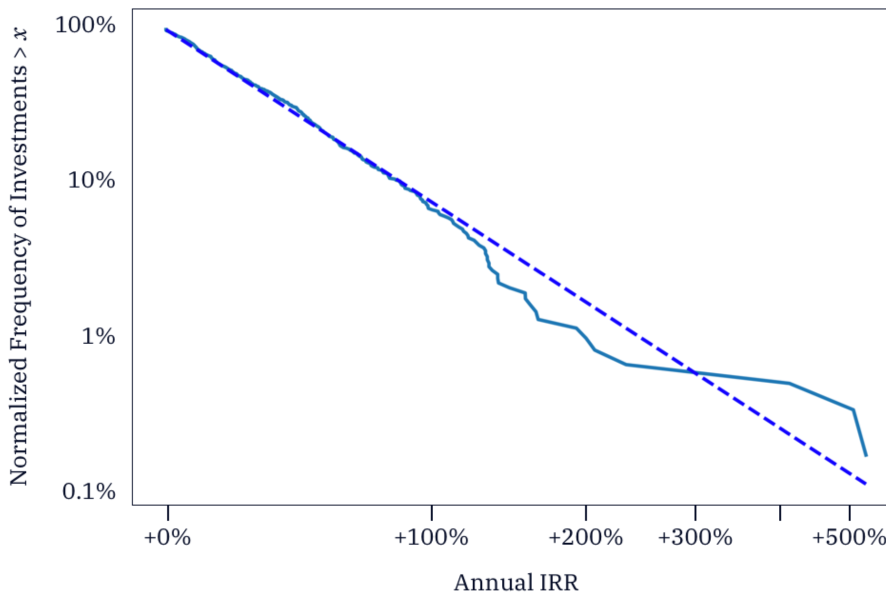


**Fig. 2.** Our first novel empirical result is that the IRRs of winning venture investments (solid blue line) are consistent with a power-law distribution (dotted blue line). Existing models from the literature have suggested that IRRs should have light tails, which would manifest as a steep, cliff-like dropoff on this kind of log-log plot. The distribution of IRRs is, empirically, a power-law fit with $\alpha = 4.8$, a relatively tame power law.

By inverting the result of Theorem 1 we obtain the following lemma that will be useful to us in later sections:

**Lemma 1.** *An investment with a return multiple drawn from a power-law distribution with shape parameter* $\alpha$ *and an effective duration of* $t$ *years has its IRR* $1 + r$ *correspondingly drawn from a power-law distribution with shape parameter*

$$A \equiv (\alpha - 1)t + 1$$

Of course, not all investments can have the same effective duration: it is tautological that a Series B investment in a company must have a shorter duration than an investment in the company's Series A that preceded it. This prompts us to investigate the relationship between IRR and duration.

## Modeling the Correlation of IRRs and Duration

Recall that in the Standard Model, investment duration and IRRs are drawn independently, implying that there should be no correlation between the two quantities. However, this is not the case from AngelList data. Figures 3 and 4 are scatterplots of IRRs against effective durations. The left figure is for the first seed investment into a company, and the right figure is for follow-on investments.

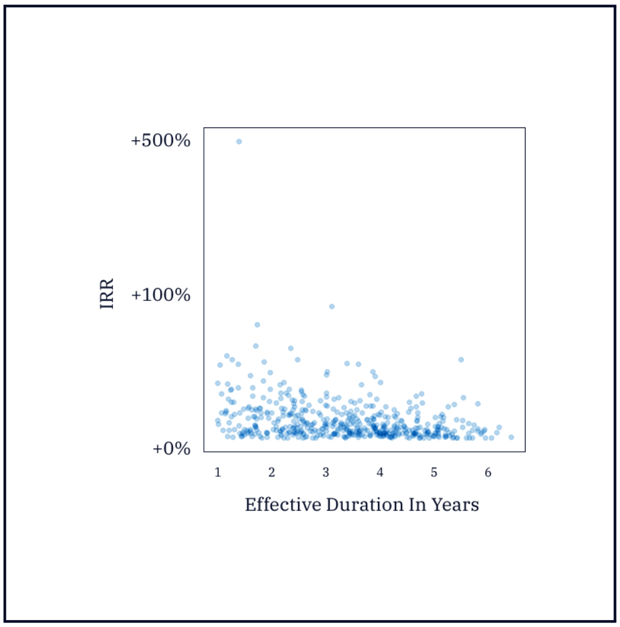


**Fig. 3.** Scatterplot of IRRs and effective durations for the first seed investment into companies.

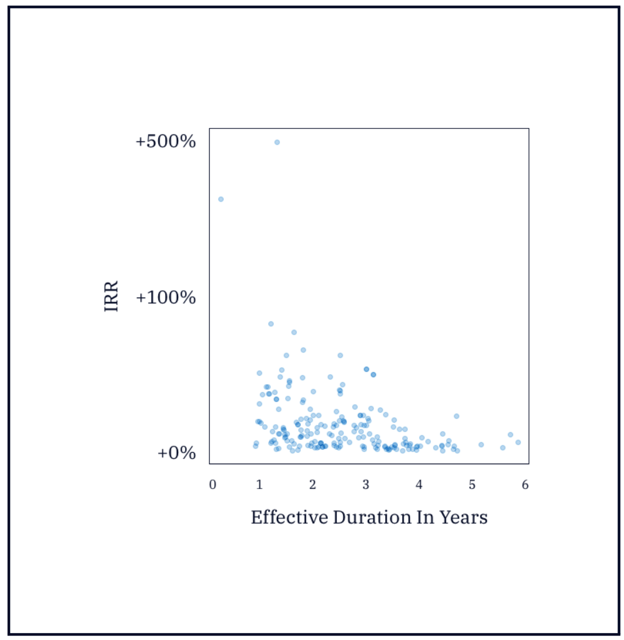


**Fig. 4.** Scatterplot of IRRs and effective durations for later, follow-on investments into companies.

Table 2 calculates the Spearman (rank-order) correlations between IRR and duration for these investments. We select the Spearman method for our correlation calculation because it is non-parametric and more robust to the outliers that arise when drawing from a power-law distribution.

Considering just the first seed investments into a company, or just the later investments, or all the investments in the dataset, the correlations are negative between effective duration and IRR. Investments that persist for a longer time have lower IRRs than shorter-duration investments.

**Table 2.** Figures 3 and 4 suggest a negative correlation between an investment's effective duration and its IRR, which negative correlation can be empirically confirmed.

| | Count | Spearman Correlation between IRR and Duration |
|---|---|---|
| First seed investments | 487 | -.32 |
| Later investments | 197 | -.40 |
| All investments | 684 | -.35 |

All of the correlations are significant at the $p < 0.001$ level.

One interpretation of this result is that *companies grow more slowly as they age*. Put another way, from a compounding returns point of view, time begins to contract for startups; their entire fifth year may only be "worth", from a compounding perspective, the same as their first six months. This interpretation leads us to devise what we call a *time contraction function*, $c(t) : \Re^+ \mapsto \Re^+$ that maps real elapsed calendar time into a functional amount of time that the company compounds their draw from the power-law distribution of IRRs into the power-law distribution of return multiples. So if $c(4) \mapsto 3$ and $c(5) \mapsto 3.5$ then a company's fifth year of existence would result in only an additional six months of time from the perspective of compounding returns. Observe that if $c(t) = t$, then there is no contraction of time and each year is as valuable as the next.

With the introduction of this time contraction function, we must consider not only the company's effective duration but also when the investment begins. As we have discussed, one of the filters that we applied to our dataset was a restriction to look only at companies where AngelList has data on their seed round; we re-index all times so that $t = 0$ is the company's first investable opportunity.

The introduction of a time contraction function appears convoluted and needlessly complex, but it has two significant advantages. The first is that it allows us to retain the original model and associated concise structure of Theorem 1 by replacing $t$ with $c(t)$ as appropriate:

**Lemma 2.** *An investment that begins at time $t$ with an effective duration of $d$ and baseline IRR drawn from a power-law distribution with shape parameter $A$ has its return multiple correspondingly drawn from a power-law* *distribution with shape parameter*

$$\alpha \equiv \frac{A + c(t+d) - c(t) - 1}{c(t+d) - c(t)}$$

Observe that we recover the initial result of Theorem 1 in the special no-contraction case of $c(t) = t$.

The second advantage of this approach is that it provides a simple way to produce the negative correlation structure that we found empirically. Take $c(t)$ to be an increasing, concave function with $c(0) = 0$ and consider the shape parameter of its power-law distribution of IRRs conditional on time $t$, $\hat{A}(t)$. From our lemmas:

$$\begin{aligned} \hat{A}(t) &= d\left(\frac{A + c(t+d) - c(t) - 1}{c(t+d) - c(t)}\right) + 1 - d \\ &= t\left(\frac{A-1}{c(t)}\right) + 1 \end{aligned}$$

Evaluating this expression at $t'' > t'$, we see that the implied distribution of IRRs will have a larger shape parameter over time:

$$\hat{A}(t'') = (A-1)\left(\frac{t''}{c(t'')}\right) + 1 > (A-1)\left(\frac{t'}{c(t')}\right) = \hat{A}(t')$$

where the inequality follows from the concavity of the contraction function. Since larger shape parameters in power laws are associated with tamer (lighter-tailed) distributions, the IRRs observed at time $t'$ will be drawing from a heavier-tailed distribution than the the IRRs observed at time $t''$, which will be made manifest in an observed negative correlation between investment durations and IRRs.

## Parameterization and Fitting

We have introduced a new model of venture capital returns, where the IRR of investments are drawn from a power-law distribution and then companies experience a contraction in the amount of time that IRR compounds into a return multiple, consistent with our observations of both power-law-distributed IRRs as well as the negative correlation between IRRs and durations. To fit our model empirically, we must solve for the appropriate time contraction function $c(t)$. There are three desiderata for the time contraction function:

1. It should be monotone non-decreasing; the number of functional years of an investment's existence for compounding purposes should not decrease in the number of actual years.
2. It should be no greater than the amount of actual time elapsed, with equality at the start of the investment period: $c(t) \leq t$ and $c(0) = 0$.
3. It should be (weakly) concave. Intuitively, it should not be the case that company growth rates accelerate over time. Mathematically, as discussed above, concavity will tend to produce the empirical negative correlation between IRRs and durations.

Following the guidance of Clauset et al. (2009) we set as our objective the maximization of the log-likelihood of our observed data. Recall that the log-likelihood of observing a value of $x$ from a power law with shape parameter $\alpha$ is:

$$ll(x|\alpha) \equiv \log(\alpha - 1) - \alpha \log x$$

Algorithm 1 solves for the log-likelihood of our data provided a contraction function $c(t)$.

**Algorithm 1:** Log-likelihood calculation

**Input:** Baseline shape parameter $A > 1$, contraction function $c : \Re^+ \mapsto \Re^+$, Dataset $\mathcal{D} = [(t_i, d_i, r_i)]$ consisting of the start, effective duration, and IRR of winning investments

**Output:** Log-likelihood $ll$ of our observed data

$ll \leftarrow 0$

**for** $(t, d, r) \in \mathcal{D}$ **do**

  $\hat{d} \leftarrow c(t + d) - c(t)$

  $\hat{\alpha} \leftarrow \frac{A + \hat{d} - 1}{\hat{d}}$

  $\hat{A} \leftarrow d\hat{\alpha} + 1 - d$

  $ll \leftarrow ll + \log\left(\hat{A} - 1\right) - \hat{A}\log(1 + r)$

**return** $ll$

For each investment, we calculate the time-contracted duration $\hat{d}$ of the investment, use that to calculate the implied power-law distribution of return multiples $\hat{\alpha}$ and then use that value to calculate the implied power-law distribution of IRRs $\hat{A}$. We then calculate the log-likelihood of observing the actual IRR $1 + r$ from this distribution. (To avoid double counting we treat $n > 1$ investments in the same company by scaling the relevant log-likelihoods by $1/n$.)

Algorithm 1 takes as input a contraction function $c(t)$. To parameterize this function we use Schumaker Shape-Preserving Interpolation (Schumaker, 1983; Judd, 1998). This approach takes Hermite interpolation data $(\vec{t}, \vec{z}, \vec{s})$ where the $t_i$ are the input values, the $z_i$ are the function values at those inputs, and the $s_i$ are the slopes at those input values. It then produces a continuously differentiable function that interpolates through the Hermite datapoints while obeying any implied convexity or concavity in those datapoints. We denote the Schumaker shape-preserving functional from $k$ input Hermite tuples as $\mathbf{S} : \Re^{3\times k} \mapsto (\Re \mapsto \Re)$.

Now our problem reduces to finding the log-likelihood maximizing parameterization of the Schumaker shape-preserving interpolation with input values that obey our desiderata. We can express this goal through the following math program:

$$\max_{A,\vec{z},\vec{s}} \; ll\left(A, \mathbf{S}(\vec{t}, \vec{z}, \vec{s})\right) \tag{1}$$

$$s.t. \tag{2}$$

$$A > 1 \tag{3}$$

$$\vec{t} = \{0, 1, 2, 4, 8\} \tag{4}$$

$$z_0 = 0 \tag{5}$$

$$s_0 \leq 1 \tag{6}$$

$$z_{i+1} \geq z_i \tag{7}$$

$$s_i \geq 0 \tag{8}$$

$$s_i \geq \left(\frac{z_{i+1} - z_i}{t_{i+1} - t_i}\right) \geq s_{i+1} \tag{9}$$

We set the input values for the Schumaker interpolation arbitrarily, but with an eye towards our distribution of durations, at $t = \{0, 1, 2, 4, 8\}$ years in Line (4). We then implement our desiderata using constraints. Line (5) sets the fixed point $c(0) = 0$. Line (6) caps the instantaneous rate of time contraction to be at most the actual time elapsed. Lines (7) and (8) are monotonicity conditions, and Line (9) ensures weak concavity. Observe that the denominators in the Line (9) constraints $t_{i+1} - t_i$ are constant scalars and not variables because we have *a priori* fixed the input values $t_i$.

The overall optimization problem is ten dimensional, solving for the function values $z_1, \ldots, z_4$, the slopes $s_0, \ldots, s_4$, and the baseline shape parameter $A$.

# Results

Figure 5 shows a plot of the optimal time contraction function. The log-likelihood-maximizing solution we found sets the corresponding baseline shape parameter $A = 1.54$.

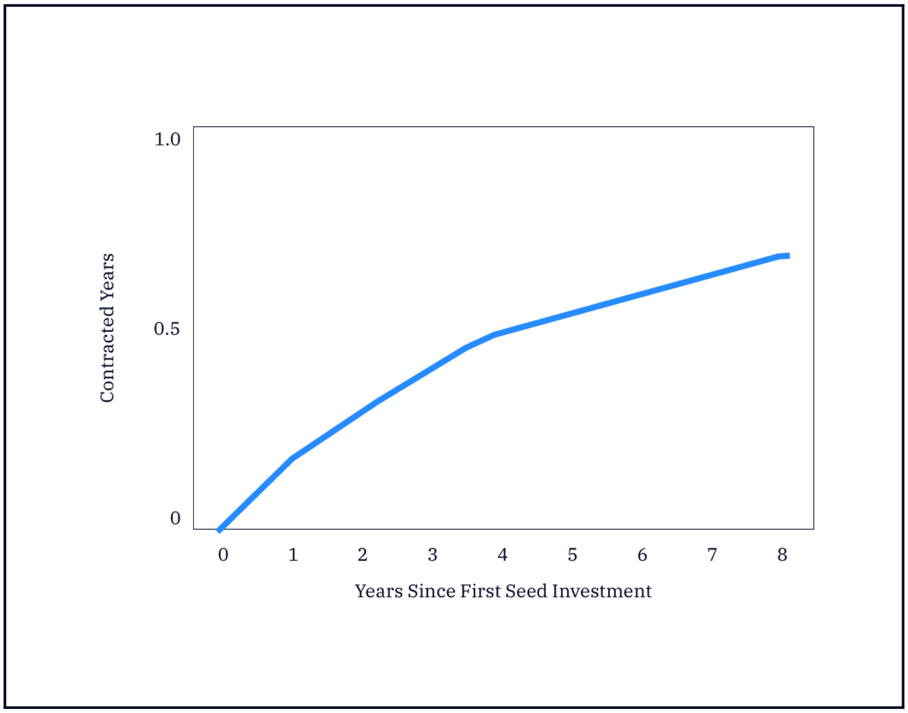


**Fig. 5.** In order to fit our empirical data that suggests a negative correlation between investment duration and IRR we introduced the theoretical concept of a time contraction function which maps real calendar time to effective time for the purposes of investment value compounding. We imposed certain constraints, like monotonicity and concavity, on that function in order to match our intuitions and avoid overfitting. This figure shows the optimal contraction function that fits our data while satisfying our constraints.

This solution may appear surprising as both the contraction function and baseline shape parameter are both much smaller in numerical magnitude than would be anticipated. However, observe that the return multiple formula from Lemma 2, and therefore our log-likelihood optimization, is nearly homogeneous in $A$ and the contraction function. Consequently, Figure 6 is a more meaningful depiction of our findings than the particular quantitative parameterization we found to be optimal. It shows the compounding value of each successive year relative to the company's first year after its seed round by calculating successive differences in the time contraction function.

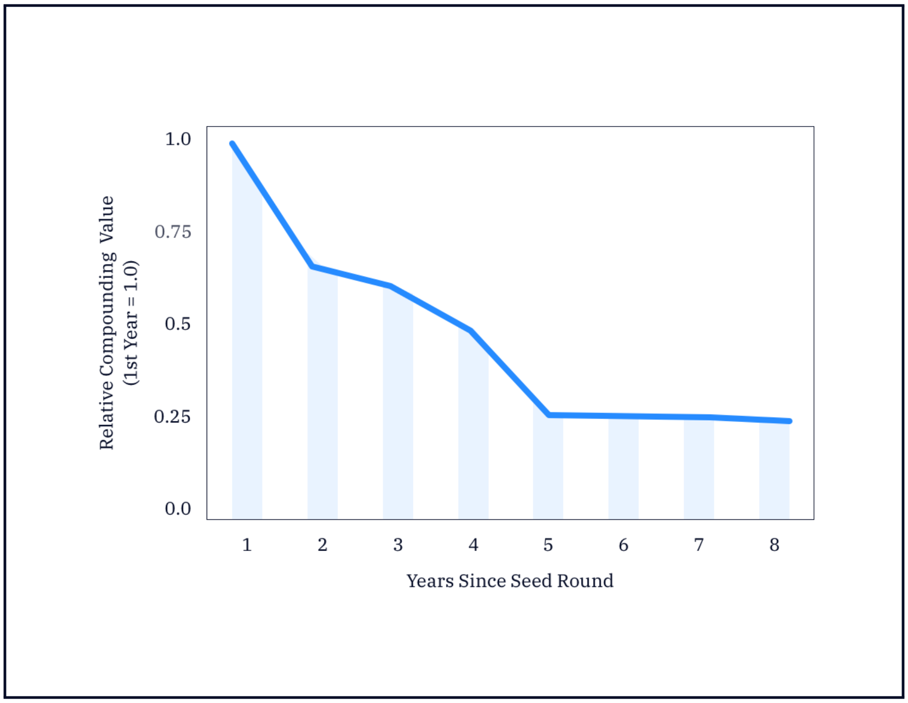


**Fig. 6.** Our results suggest that each successive year of a company's existence is worth less and less from the perspective of compounding investment growth. For instance, we can expect a company to grow about as much in years three and four, or in years four, five, and six, as it does in its first year.

Figure 6 suggests that a company's rate of growth drops off quickly following its first year. The fifth year of a company's existence as an investable entity is worth, from the perspective of valuation compounding, about one-quarter as much as its first year and less than half of its second year.

While it is evident from Figure 6 that only earliest years of a company's existence tend to have the highest growth rates, companies still tend to continue growing in their later years. Figure 7 shows the compounding effects of that growth on the return multiples of a seed investment made at time $t = 0$.

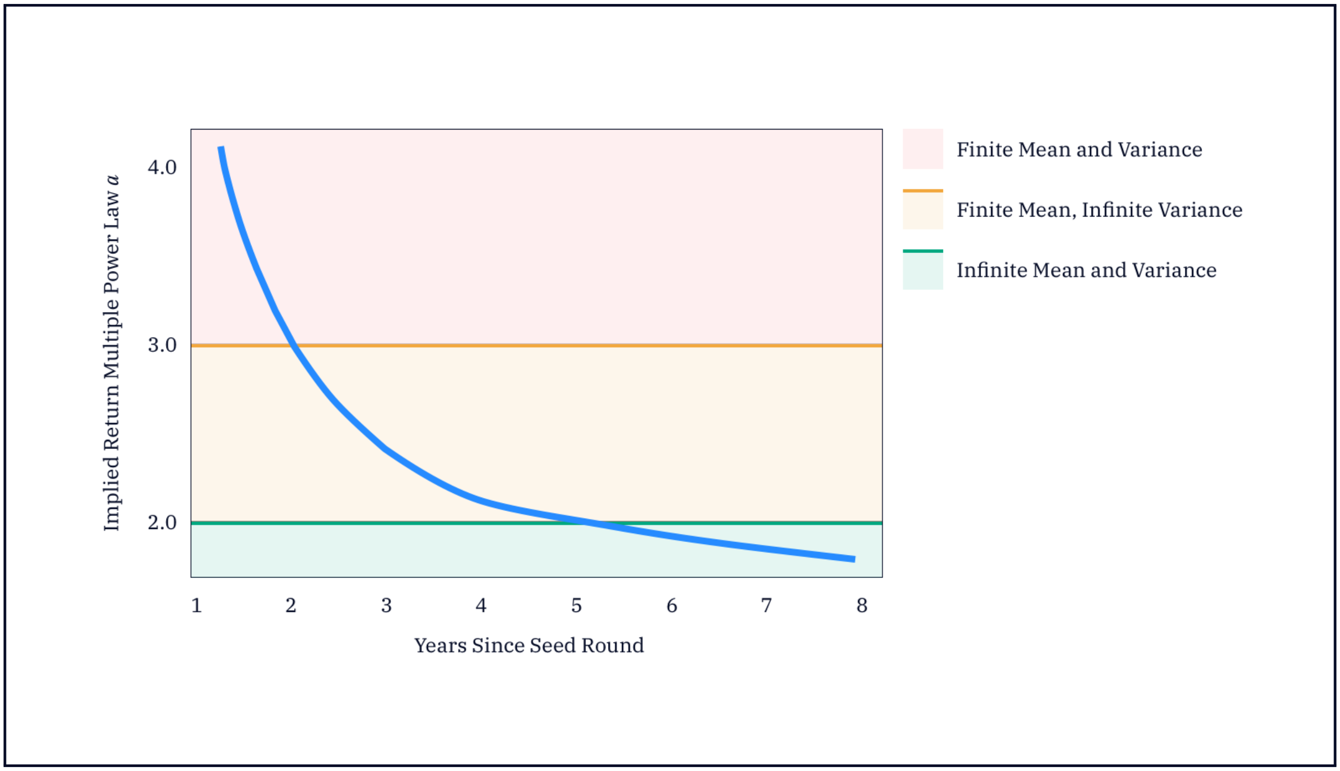


**Fig. 7.** Because companies continue growing, the shape parameter of the power-law distribution of return multiples of a seed round investment continues to decrease, producing more extreme return multiples, over time. After 5.1 years the investment's return multiple begins to draw from an $\alpha < 2$ power law with unbounded mean. The lines at $\alpha = 2$ and 3 represent breakpoints in the qualitative properties of power-law distributions and are drawn for reference.

In Figure 7 we considered what happens to the return multiples of a seed investment as its effective duration lengthens. We also want to consider the inverse question: what does the distribution of return multiples look like for winning investments made at various points in time for a fixed time of exit? Figure 8 shows the shape parameter of return multiples over time for investments that exit eight years after they have been made.

Our interpretation of Figure 8 is that seed investments, at least at an eight year exit horizon, are relatively unique in drawing from an $\alpha < 2$ power law and that this property does not extend for that long in the company's investable life. Series A and perhaps Series B rounds, made one or two years after the seed round, would draw from an $\alpha$ between 2 and 3, implying unbounded variance but not unbounded mean. In this eight-year exit example after about year three, so perhaps the Series B or Series C as well as later rounds, the investments are drawing from an $\alpha > 3$ power law where variance is finite and the Central Limit Theorem

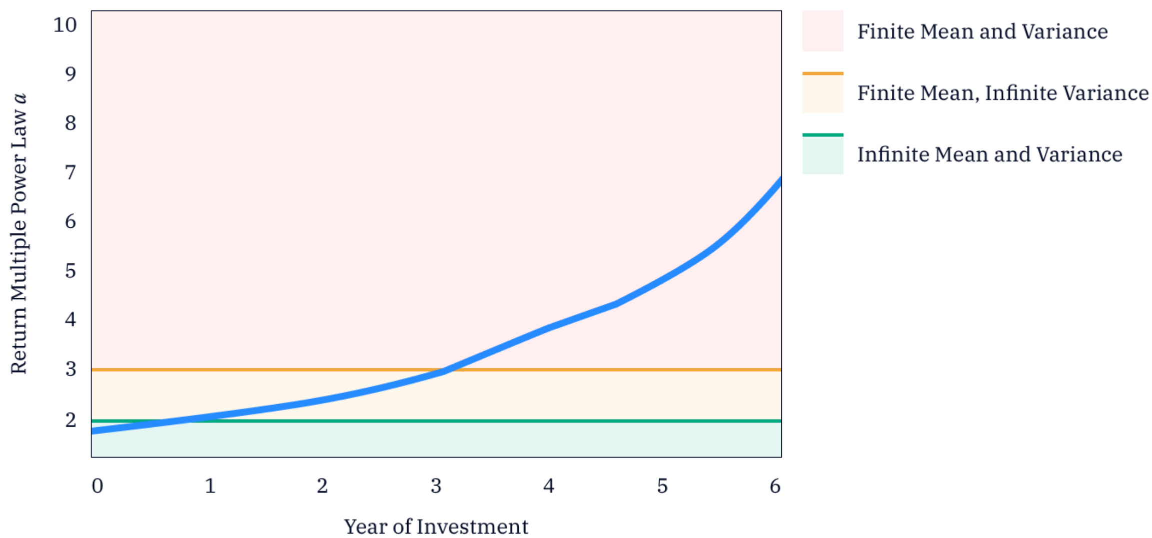


**Fig. 8.** The shape parameters for the power-law distributions of return multiples for companies that exit eight years after their first investable opportunity based on when investors participate. Only the earliest investment opportunities are associated with the lowest, and therefore most extreme, distributions of returns. The black at $\alpha = 2$ and 3 represent breakpoints in the qualitative properties of power-law distributions and are drawn for reference.

holds. For very late stage investments, say Series E rounds made six years after the company's seed round, the suggested power law of return multiples is very tame, to the point that it is probably unidentifiable as a power law versus a volatile log-normal distribution, especially when considering the necessarily small sample size of companies that achieve such an outcome. We suggest that this effect squares our results with how Gornall and Strebulaev (2019) can comfortably advance a log-normal distribution of forward returns for the very-late-stage companies they consider.

## Discussion

Throughout our analysis we have ignored money-losing investments, but we will now address them directly. It does not follow automatically that the presence of enormous outlier returns make an asset class a good investment. A large multi-state Powerball lottery can produce a return of several million times the cost of the winning ticket, but that return is more than offset by the (exponentially larger) pool of losing tickets. However, as long as a constant fraction of investments lose money, money-losing investments will only affect the quantitative, rather than qualitative, properties of the investment class. This seems to be the case in venture capital as currently constituted; roughly half of the venture investments that we studied lose money.

We have contextualized money-losing investments in this way because in the presence of an $\alpha < 2$ power law for returns the opportunity cost of missing a single winning investment is theoretically infinite, and investors without perfect foresight that select from a static pool of potential investments achieve their highest expected return by investing in every seed-stage deal. However, we do not feel that investing in everything is practical. We worry that an investor promising to blindly fund every whisper of a new company would fundamentally alter the investment universe they are exposed to by introducing a huge number of new money-losing investments that otherwise would not have been created but for the investor's universal funding policy. Consequently, our results suggest that at the seed stage investors should put money into every investment that clears some minimum threshold.[3]

Our research provides a principled quantitative underpinning for a broadly indexed "spray and pray" model of investing at a company's earliest stages. This is a controversial and contrarian point of view because this kind of investing has been maligned, or at least misunderstood, by traditional venture capitalists. As an example of this point of view, in 2006 venture capitalist Bryce Roberts wrote that:

> The heart of the venture business is being able to separate signal from noise, rolling up your sleeves and working with the entrepreneur to build something successful. Is there any coincidence that Yahoo, Google, YouTube and so many of these "hits" have come from a handful of firms? Certainly timing and luck play a huge part in our business, but minimizing it to a seat at a roulette table is a shame.[4]

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[3] Calibrating and quantifying the investment threshold is a topic for future work. As a workable definition one might consider something like "Is there a version of this company that could raise a Series A from a VC?"
[4] Quoted in http://radar.oreilly.com/2006/11/what-vcs-learned-from-odeo-and.html

This is also a point of view that sits well with the institutional investors and investment advisors that are the limited partners of venture capital funds. A VC who pitches their careful diligence, hard work, and sophisticated criteria for investment selection seems like a much better steward of an LP's money than a VC who pitches making so many early-stage investments that they could not keep up with their companies even if they tried.

We ground our argument for broadly indexing in the idea that the opportunity cost of missing a winning investment is theoretically infinite, but this only holds at the seed round. Our results suggest that, speaking generally at Series A and perhaps at Series B there can be desirable portfolio-level effects from making more investments (for instance, that the expected median return of a portfolio increases in the number of investments made). But our results also suggest that the opportunity cost for missing a winning investment in these later rounds is bounded; at later stages companies experience a shorter duration of lower compounding than at their Seed round. Consequently, it is entirely appropriate that later-stage investors should reject the "spray and pray" idea and be thoughtful and discerning when they participate, but also that those same investors are making an error when they take that restrained and nuanced approach to investing in the earliest stages of a company.

This difference in approaches aligns with one of our most intriguing results: that over time, the fundamental nature of the stock of startup companies changes, so that winning investors in a Series D round can expect very different returns than the winning investors in the Seed round of that same company, to the point that the two investors are in a qualitative sense investing in different asset classes. This result explains why VCs partition themselves primarily by company stage; there is no firm that will make new fintech investments from Seed to Series F, but there are many VC firms that will do the Seed round of a hardware, a biotech, and a fintech company. (Consider the intra-corporation split between GV, the entity formerly known as Google Ventures, which advertises their diverse early-stage investments in "Consumer", "Enterprise", "Life Science", and "Frontier Tech" companies, and CapitalG, the entity formerly known as Google Capital, which does later-stage investing.)

One of the implications of this result for policymakers is that there should be lessened friction for early-stage investors to cash out of their winning investments. Those investors signed up for a certain asset class, and after several years, have found themselves owning a different asset class with different risks and returns. Our results also suggest that those investors could potentially earn higher rates of return by reinvesting their money into a wise selection of new early-stage investments.

Another reason that there is a need for liquidity for early-stage investors is that companies are now staying private longer (Erdogan et al., 2016). Our perception is that there is a vague uneasiness about this recent change to the industry from policymakers and the general public. Our results suggest a mathematical underpinning for this concern: that companies staying private longer have created an asset class drawing returns from an $\alpha < 2$ power law entirely outside the scope of individual retail investors. Even starting from a very small investment base (say, a few million dollars invested at the seed round of an individual company), the top winning investments that are drawing from this distribution will result in perhaps tens of billions of dollars of created wealth. In fact, one of the challenging and unintuitive properties of an $\alpha < 2$ power law for returns is that it does not matter if it starts from a small capital base; on a long enough time span an asset class drawing from such a distribution will eventually capture an arbitrarily large fraction of created wealth. We argue from equity and fairness that if a sizable fraction of the wealth created by an economy will be generated from early-stage investing, then a correspondingly large fraction of society should have the ability to participate in that wealth creation. That is why we call on policymakers to take steps that would enable retail investors to broadly gain access to early-stage venture capital as an asset class within their portfolios, consistent with the appropriate investor protections.[5] To be clear: we are not advocating for an expansion of access to *individual* early-stage investments for non-accredited, retail investors. Early-stage investing remains incredibly risky, particularly for investors who lack the means to appropriately diversify. Individual investments regularly lose the entirety of their principal, and the typical retail investor on their own simply does not see the volume and quality of deals that can result in the outlier returns that underpin the analysis presented in this paper. But policymakers should be concerned that changes in the structure of private investing have created a powerful wealth-generating mechanism entirely out of the hands of the investing public.

[5] For example, we outlined certain steps that we believe policymakers could take to provide investors with increased access to diversified startup investment opportunities through pooled investment vehicles in our response to the SEC's Concept Release on Harmonization of Securities Offering, File No. S7-08-19, available at https://www.sec.gov/comments/s7-08-19/s70819-6203757-192567.pdf

# Proof of Theorem 1

**Theorem 1.** *An investment with its IRR $1+r$ drawn from a power-law distribution with shape parameter $A$ and an effective duration of $t$ years has its return multiple correspondingly drawn from a power-law distribution with shape parameter*

$$\alpha \equiv \frac{A+t-1}{t}$$

*Proof.* Consider expressing IRR $1+r$ in terms of the return multiple $m$. From Definition 4:

$$m = (1+r)^t \to (1+r) = m^{1/t}$$

Now let $f_M$ be the density function of return multiples and $f_R$ be the density function of IRRs. Then by the rule of probability transforms:

$$f_M(m) = f_R(m^{1/t}) \left| \frac{d}{dm} \left(m^{1/t}\right) \right| = f_R(m^{1/t}) \left( \frac{1}{t} m^{\frac{1-t}{t}} \right)$$

Since $f_R$ is a power-law distribution with shape parameter $A$:

$$f_R(m^{1/t}) = (A-1) \left(m^{1/t}\right)^{-A} = (A-1) \left(m^{-A/t}\right)$$

Simplifying:

$$\begin{aligned} f_M(m) &= f_R(m^{1/t}) \left( \frac{1}{t} m^{\frac{1-t}{t}} \right) \\ &= (A-1) \left(m^{-A/t}\right) \left( \frac{1}{t} m^{\frac{1-t}{t}} \right) \\ &= \frac{A-1}{t} m^{\frac{1-t-A}{t}} \\ &= \left( \frac{A+t-1}{t} - 1 \right) m^{-\frac{A+t-1}{t}} \end{aligned}$$

Which is a power-law distribution with shape parameter $\frac{A+t-1}{t}$. □

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Returns presented and discussed above were calculated based on the methodology described above and exclude certain types of AngelList investments. These data do not include investments by pooled investment vehicles such as angel funds, proprietary platform funds, and other types of AngelList funds. For purposes of calculating unrealized returns of AngelList funds, companies are valued with industry-standard methods, and our valuations have not been audited by a third-party.

Valuations are generally marked to a company's latest priced financing round, as disclosed to us. While our valuation sources are believed to be reliable, we do not undertake to verify the accuracy of such valuations. Companies that have not received new investments in a priced round since the last mark are held at cost or may be marked down at our discretion according to our valuation policy. Valuations and returns do not account for liquidation preferences and other non-financial terms that may affect returns. Investments in later-stage companies may be sent to a third-party for valuation if (i) the company's estimated value is over \$100M, (ii) the investment is estimated to be worth over \$10M and (iii) 24 months have passed since the last investment.